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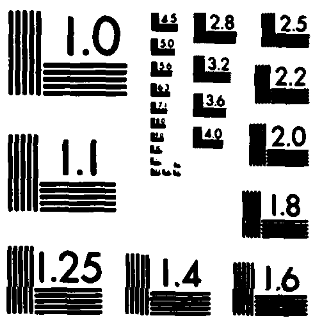
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A BOUNDARY VALUE PROBLEM FOR QUASILINEAR DEGENERATE PARABOLIC EQUATIONS

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ABSTRACT

It is shown that the arguments developed in Vol'pert and Hudjaev's paper for the Cauchy problem and in the author's paper for the first boundary value problem can be extended to other kinds of boundary value problems. As an example, equations of the form

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(a(t,x,u)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial x}f(t,x,u) + g(t,x,u) \qquad (a(t,x,u) \geq 0)$$

with the boundary conditions

$$a\frac{\partial u}{\partial x} + f = 0 \qquad (x = 0)$$

$$u = 0 \qquad (x = 1)$$

and the initial condition

$$u = u_0(x) \qquad (t = 0)$$

are investigated, and the existence, uniqueness and continuous dependence on the initial value of generalized solutions are proved under certain conditions. In proving the existence, the key step is to establish estimates on solutions u_ε of regularized problem, especially the uniform estimate of

$$\left|\frac{\partial u_\varepsilon}{\partial t}\right|_{L^1} \quad \text{and} \quad \left|\frac{\partial u_\varepsilon}{\partial x}\right|_{L^1}.$$

AMS (MOS) Subject Classifications: 35K60, 35K65

Key Words: quasilinear degenerate parabolic equations, nonlinear boundary value problems, regularization, existence, uniqueness and continuous dependence

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SIGNIFICANCE AND EXPLANATION

Using the theory of functions of bounded variation, Vol'pert and Hudjaev successfully treated the initial-value problem for a class of degenerate parabolic equations in one space dimension. Of particular interest was their ability to incorporate even the "completely degenerate" case of a scalar conservation law in the class they treated. The author subsequently treated the first boundary value problem in a similar spirit and generality. The current work shows that analogous results can be obtained for other boundary conditions. As before, regularization is used to obtain existence results for approximate problems. New estimates are obtained on the approximations which allow passage to the limit.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

A BOUNDARY VALUE PROBLEM FOR QUASILINEAR DEGENERATE PARABOLIC EQUATIONS

Zhuoqun Wu

§1. Introduction

For quasilinear equations of the form

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(a(t,x,u)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial x} f(t,x,u) + g(t,x,u) \tag{1.1}$$

with

$$a(t,x,u) \geq 0 ,$$

the Cauchy problem has been investigated in [1] and the first boundary value problem in [2], [3]. In this paper, we will show that the arguments developed in [1]-[3] can be extended to other kinds of boundary value problems. As an example we consider (1.1) with the boundary conditions

$$(1.2)_0 \qquad a(t,x,u)\frac{\partial u}{\partial x} + f(t,x,u) = 0 \qquad (x = 0)$$

$$(1.2)_1 \qquad u = 0 \qquad (x = 1)$$

and the initial condition

$$u = u_0(x) \qquad (t = 0) . \tag{1.3}$$

For simplicity, only homogeneous boundary conditions are dealt with here.

Let $Q_T = (0,T) \times (0,1)$. Assume that the functions a, f and g are smooth for $(t,x) \in \bar{Q}_T$ and $u \in R$.

The problem will be formulated in a generalized sense as follows (with notations referred to [2]).

<u>Definition</u>. A function $u \in L^\infty(Q_T) \cap BV(Q_T)$ is called the generalized solution of problem (1.1), (1.2), (1.3), if the following conditions are fulfilled:

1) There exists a function $g \in L^2(Q_T)$, such that

$$\iint_{Q_T} \phi g\, dt dx = \iint_{Q_T} \phi\,\widehat{a(t,x,u)}\,\frac{\partial u}{\partial x}\, dt dx \qquad \forall \phi \in C_0^\infty(Q_T) \tag{1.4}$$

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where $r(t,x,u) = \sqrt{a(t,x,u)}$ and $\widehat{r(t},x,u)$ denotes the mean value of composition of $r(t,x,u)$ and $u(t,x)$.

2) u satisfies the integral inequality

(1.5) $$\iint_{Q_T} \operatorname{sgn}(u - k)[(u - k) \frac{\partial \phi}{\partial t} - (\widehat{a(t},x,u) \frac{\partial u}{\partial x} + f(t,x,u) - f(t,x,k)) \frac{\partial \phi}{\partial x} +$$

$$+ (f_x(t,x,k) + g(t,x,u))\phi]dtdx \geq 0 \qquad \forall \phi \in C_0^\infty(Q_T), \quad \phi \geq 0 .$$

3) For almost all $t \in [0,T]$,

$(1.6)_0$ $$\gamma(\widehat{a(t},x,u) \frac{\partial u}{\partial x} + f(t,x,u)) = 0 \qquad (x = 0)$$

$(1.6)_1$ $$\gamma u = 0 \qquad (x = 1) .$$

4) For almost all $x \in [0,1]$,

(1.7) $$\gamma u = u_0(x) \qquad (t = 0) .$$

<u>Remark</u>. From (1.4) it follows that the measure $\widehat{r(t},x,u) \frac{\partial u}{\partial x}$ - and hence the measure $\widehat{a(t},x,u) \frac{\partial u}{\partial x}$ - is absolutely continuous. (1.5) implies, in particular, that

$$\iint_{Q_T} (u \frac{\partial \phi}{\partial t} - (\widehat{a(t},x,u) \frac{\partial u}{\partial x} + f(t,x,u)) \frac{\partial \phi}{\partial x} + g(t,x,u)\phi)dtdx = 0 \qquad \forall \phi \in C_0^\infty(Q_T) .$$

Hence $\frac{\partial}{\partial x} \left(\widehat{a(t},x,u) \frac{\partial u}{\partial x}\right)$ is a measure and the trace $\gamma\left(\widehat{a(t},x,u) \frac{\partial u}{\partial x}\right)$ at the boundary points exists.

We will first prove the uniqueness and stability of generalized solutions (§2) and then study the existence (§3, §4). Similar to [1]-[3], we prove the existence by means of the method of parabolic regularization, namely, consider

$(1.1)^\varepsilon$ $$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left((a(t,x,u) + \varepsilon) \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial x} f(t,x,u) + g(t,x,u) \qquad (\varepsilon > 0)$$

instead of (1.1) and

$(1.2)_0^\varepsilon$ $$(a(t,x,u) + \varepsilon) \frac{\partial u}{\partial x} + f(t,x,u) = 0 \qquad (x = 0)$$

instead of $(1.2)_0$. However, it is somewhat difficult to obtain some of the estimates on the solutions of regularized problems $(1.1)^\varepsilon$, $(1.2)_0^\varepsilon$, $(1.2)_1$, (1.3), which we need for the proof of existence.

§2. Uniqueness and Stability

__Theorem 1__ (Stability of generalized solutions). Let u,v be generalized solutions of equation (1.1) with the boundary conditions (1.2) and initial conditions

$$\gamma u = u_0, \qquad \gamma v = v_0 \qquad (t = 0) .$$

Then for almost all $t \in [0,T]$, we have

$$\int_0^1 |u(t,x) - v(t,x)|dx \le e^{Nt} \int_0^1 |u_0(x) - v_0(x)|dx ,$$

where

$$N = \sup_{Q_T} |\int_0^1 g_u(t,x,\lambda u_1(t,x) + (1-\lambda)u_2(t,x))d\lambda| .$$

As a consequence, we have

__Theorem 2__ (Uniqueness of generalized solutions). The boundary value problem (1.1), (1.2), (1.3) has at most one generalized solution.

__Proof of Theorem 1__. For any nonnegative function $\phi \in C_0^\infty(Q_T)$, we have (see [1])

$$\iint_{Q_T} \operatorname{sgn}(u - v)[(u - v)\frac{\partial\phi}{\partial t} - (\widehat{a(t,x,u)}\frac{\partial u}{\partial x} - \widehat{a(t,x,v)}\frac{\partial v}{\partial x} + f(t,x,u) - f(t,x,v))\frac{\partial\phi}{\partial x} + \tag{2.1}$$

$$+ (g(t,x,u) - g(t,x,v))\phi]dtdx \ge 0 .$$

Since $\gamma u = \gamma v = 0$ $(x = 1)$, this inequality holds even for those nonnegative functions $\phi \in C^\infty(\bar{Q}_T)$ with $\operatorname{supp}\phi \subset (0,T) \times (0,1]$ (see [2]).

Let

$$\rho_h(\sigma) = \int_{-\infty}^{\sigma} \delta_h(\tau)d\tau = \frac{1}{h}\int_{-\infty}^{\sigma} \delta(\frac{\tau}{h})d\tau \qquad (h > 0) \tag{2.2}$$

where $\delta(\sigma) \in C^\infty(R)$, $\delta(\sigma) \ge 0$, $\delta(\sigma) = 0$ $(|\sigma| \ge 1)$, $\int_{-\infty}^{\infty} \delta(\tau)d\tau = 1$.

Taking $\phi = \rho_h(x - 2h)\psi(t)$ with $\psi \in C_0^\infty(0,T)$ and $\psi \ge 0$, from (2.1) we obtain

$$\iint_{Q_T} \operatorname{sgn}(u - v)[(u - v)\rho_h(x - 2h)\psi' -$$

$$- (\widehat{a(t,x,u)}\frac{\partial u}{\partial x} - \widehat{a(t,x,v)}\frac{\partial v}{\partial x} + f(t,x,u) - f(t,x,v))\delta_h(x - 2h)\psi +$$

$$+ (g(t,x,u) - g(t,x,v))\rho_h(x - 2h)\psi]dtdx \ge 0 .$$

Letting $h \to 0$ and using Lemma 2 in [2] yield

$$\iint_{Q_T} |u - v|\psi' dtdx > -\int_0^T \operatorname{sgn}(\gamma u - \gamma v)[\gamma(\widehat{a(t},x,u) \frac{\partial u}{\partial x} + f(t,x,u)) -$$

$$- \gamma(a(t,x,v) \frac{\partial v}{\partial x} + f(t,x,v))]_{x=0} \psi dt -$$

$$- \iint_{Q_T} \operatorname{sgn}(u - v)(g(t,x,u) - g(t,x,v))\psi dtdx .$$

Hence, by virtue of the boundary condition $(1.6)_0$,

$$\iint_{Q_T} |u - v|\psi' dtdx > - \iint_{Q_T} \operatorname{sgn}(u - v)(g(t,x,u) - g(t,x,v))\psi dtdx .$$

Let $0 < s < \tau < T$ and take $\psi(t) = \rho_h(t - s) - \rho_h(t - \tau)$. Then

$$\iint_{Q_T} |u - v|(\delta_h(t - s) - \delta_h(t - \tau)dtdx$$

$$> \iint_{Q_T} \operatorname{sgn}(u - v)(g(t,x,u) - g(t,x,v))(\rho_h(t - s) - \rho_h(t - \tau)dtdx .$$

As $h \to 0$, this gives

$$\int_0^1 |u(\tau,x) - v(\tau,x)|dx < \int_0^1 |u(s,x) - v(s,x)|dx + N \int_s^\tau \int_0^1 |u(t,x) - v(t,x)|dtdx .$$

Hence, by Gronwall's Lemma we obtain

$$\int_0^1 |u(\tau,x) - v(\tau,x)|dx < e^{N(\tau-s)} \int_0^1 |u(s,x) - v(s,x)|dx$$

and the desired result follows by letting $s \to 0$.

§3. Estimates on $\{u_\varepsilon\}$

In addition to the smoothness condition on functions a, f and g, we assume that $f(t,0,0) \equiv 0$ for $t \in [0,T]$, $f_u(t,0,u) \leq 0$ for $t \in [0,T]$, $u \in R$ and $f_{xu} + g_u$ is bounded above for $(t,x) \in \bar{\Omega}_T$, $u \in R$ and that compatibility conditions are satisfied so that for any $\varepsilon > 0$, the regularized problem $(1.1)^\varepsilon$, $(1.2)_0^\varepsilon$, $(1.2)_1$, (1.3) has a solution $u_\varepsilon \in C^2(\bar{\Omega}_T) \cap C^3(\Omega_T)$.

<u>Proposition 1</u> (Maximum principle). Let $m = \max_{x\in[0,1]} |u_0|$, $m_1 = \sup(f_{xu} + g_u)$, $m_2 = \max_{\bar{\Omega}_T} |f_x(t,x,0) + g(t,x,0)|$ and $\lambda > m_1$. Then

$$|u_\varepsilon(t,x)| \leq \max\{me^{\lambda t}, \frac{m_2 e^{\lambda t}}{\lambda - m_1}\} \quad \text{for} \quad (t,x) \in \Omega_T . \tag{3.1}$$

In particular, we have

$$|u_\varepsilon| \leq M \quad \text{for} \quad (t,x) \in \Omega_T \tag{3.2}$$

with a contant M independent of ε.

<u>Proof</u>. Let

$$u = u_\varepsilon = (w + k)e^{\lambda t}, \qquad w = -k + ue^{-\lambda t}$$

where $k = \max(m, \frac{m_2}{\lambda - m_1})$. Then

$$\frac{\partial w}{\partial t} - (a + \varepsilon)\frac{\partial^2 w}{\partial x^2} - (a_x + a_u e^{\lambda t}\frac{\partial w}{\partial x} + f_u e^{\lambda t})\frac{\partial w}{\partial x} + (\lambda - \widetilde{f_{xu} + g_u})w + \tag{3.3}$$

$$+ (\lambda - \widetilde{f_{xu} + g_u})k - (f_x(t,x,0) + g(t,x,0))e^{-\lambda t} = 0 ,$$

where $\widetilde{f_{xu} + g_u}$ denotes the value of $f_{xu} + g_u$ at some point.

We want to prove $w \leq 0$. If it is not true, then there exists a point (t_0,x_0) with $0 < t_0 \leq T$, $0 \leq x_0 \leq 1$, such that $w(t_0,x_0) > 0$, $w(t_0,x_0)$ being the maximum of w on $\bar{\Omega}_T$. We can prove that at (t_0,x_0),

$$\frac{\partial w}{\partial t} \geq 0, \quad \frac{\partial w}{\partial x} = 0, \quad \frac{\partial^2 w}{\partial x^2} \leq 0 , \tag{3.4}$$

which contradicts (3.3). Obviously, (3.4) holds if $0 < x_0 < 1$. By $(1.2)_1$, $x_0 = 1$ is

impossible. If $x_0 = 0$ and $\frac{\partial w}{\partial x}(t_0,0) < 0$, then $\frac{\partial u}{\partial x}(t_0,0) < 0$. Since $u(t_0,0) > 0$ and $f(t,0,0) \equiv 0$, $f_u(t,0,u) < 0$ by assumption, we have

$$(a + \varepsilon)\frac{\partial u}{\partial x} + f = (a + \varepsilon)\frac{\partial u}{\partial x} + \tilde{f}_u u < 0$$

at $(t_0,0)$, which contradicts $(1.2)_0^\varepsilon$. So (3.4) holds even if $x_0 = 0$.

Thus we have proved that $w \le 0$, i.e. $u \le ke^{\lambda t}$. Similarly, we can prove that $-ke^{\lambda t} \le u$ by setting $u = u_\varepsilon = (w - k)e^{\lambda t}$ instead of $u = u_\varepsilon = (w + k)e^{\lambda t}$. The proof is complete.

<u>Proposition 2</u>.

$$\iint_{Q_T} a(t,x,u_\varepsilon)\left(\frac{\partial u_\varepsilon}{\partial x}\right)^2 dtdx \le M_1 \tag{3.5}$$

with a constant M_1 independent of ε.

<u>Proof</u>. Multiply $(1.1)^\varepsilon$ by u_ε, integrate over Q_T and use $(1.2)_0^\varepsilon$ and (3.2).

<u>Proposition 3</u>. Assume that

1^0. $f_u < 0$ for $(t,x) \in \bar{Q}_T$, $u \in E_0$ or $f_u > 0$ for $(t,x) \in \bar{Q}_T$, $u \in E_0$, where $E_0 = \{u;\ a(t,x,u) = 0$ for some $(t,x) \in \bar{Q}_T\}$;

2^0. $\frac{a_t}{a + \varepsilon} = p$ itself and its derivatives are bounded uniformly in ε for $(t,x) \in \bar{Q}_T$ and u on any finite interval.

Then

$$\int_0^1 \left|\frac{\partial u_\varepsilon}{\partial t}\right| dx \le M_2, \quad \iint_{Q_T} \left|\frac{\partial x_\varepsilon}{\partial x}\right| dtdx \le M_3 \tag{3.6}$$

with constants M_2, M_3 independent of ε.

<u>Proof</u>. Let $v = \frac{\partial u_\varepsilon}{\partial t}$, $w = \frac{\partial u_\varepsilon}{\partial x}$. Differentiate $(1.1)^\varepsilon$ with respect to t and multiply the resulting relation by $\mathrm{sgn}_\eta(v)$:

$$\begin{aligned}\frac{\partial I_\eta(v)}{\partial t} = {} & \frac{\partial}{\partial x}\{\mathrm{sgn}_\eta(v)\frac{\partial}{\partial t}[(a + \varepsilon)w + f]\} - \mathrm{sgn}_\eta'(v)(a + \varepsilon)\left(\frac{\partial v}{\partial x}\right)^2 \\ & - \mathrm{sgn}_\eta'(v)\frac{\partial v}{\partial x}(a_t + a_u v)w - \mathrm{sgn}_\eta'(v)\frac{\partial v}{\partial x}(f_t + f_u v) + \mathrm{sgn}_\eta(v)(g_t + g_u v)\end{aligned} \tag{3.7}$$

where

$$I_\eta(\tau) = \int_0^\tau \mathrm{sgn}_\eta(s)ds, \qquad \mathrm{sgn}_\eta(\tau) = \begin{cases} -1 & (\tau < -\eta) \\ \frac{\tau}{\eta} & (|\tau| \le \eta) \\ 1 & (\tau > \eta) . \end{cases}$$

Two of the terms on the right of (3.7), $-\mathrm{sgn}_\eta'(v)\frac{\partial v}{\partial x}a_t w$ and $-\mathrm{sgn}_\eta'(v)\frac{\partial v}{\partial x}f_t$, need to be treated further. Expressing a_t as $(a+\varepsilon)p$ and using $(1.1)^\varepsilon$, we have

$$-\mathrm{sgn}_\eta'(v)\frac{\partial v}{\partial x}a_t w = -\frac{\partial}{\partial x}(\mathrm{sgn}_\eta(v)a_t w) + \mathrm{sgn}_\eta(v)\frac{\partial}{\partial x}(a_t w)$$

$$= -\frac{\partial}{\partial x}(\mathrm{sgn}_\eta(v)(a+\varepsilon)pw) + \mathrm{sgn}_\eta(v)\frac{\partial p}{\partial x}(a+\varepsilon)w + \mathrm{sgn}_\eta(v)p\frac{\partial}{\partial x}[(a+\varepsilon)w]$$

$$= -\frac{\partial}{\partial x}(\mathrm{sgn}_\eta(v)(a+\varepsilon)pw) + \mathrm{sgn}_\eta(v)\frac{\partial p}{\partial x}(a+\varepsilon)w + \mathrm{sgn}_\eta(v)pv -$$

$$- \mathrm{sgn}_\eta(v)p(f_x + f_u w) - \mathrm{sgn}_\eta(v)pg .$$

Clearly

$$-\mathrm{sgn}_\eta'(v)\frac{\partial v}{\partial x}f_t = -\frac{\partial}{\partial x}(\mathrm{sgn}_\eta(v)f_t) + \mathrm{sgn}_\eta(v)(f_{tx} + f_{tu}w) .$$

Substitute these into (3.7) and throw down the second term on the right which is nonpositive. Then

$$\frac{\partial I_\eta(v)}{\partial t} \le \frac{\partial}{\partial x}\{\mathrm{sgn}_\eta(v)\frac{\partial}{\partial t}[(a+\varepsilon)w + f]\} - \frac{\partial}{\partial x}[\mathrm{sgn}_\eta(v)(a+\varepsilon)pw] -$$

$$-\frac{\partial}{\partial x}(\mathrm{sgn}_\eta(v)f_t) - \mathrm{sgn}_\eta'(v)\frac{\partial v}{\partial x}(a_u + f_u)v + \mathrm{sgn}_\eta(v)(p + g_u)v +$$

$$+ \mathrm{sgn}_\eta(v)[\frac{\partial p}{\partial x}(a+\varepsilon) - pf_u + f_{tu}]w + \mathrm{sgn}_\eta(v)(g_t - pf_x + f_{tx} - pg) .$$

Integrate this inequality over Ω_t, use $(1.2)_0^\varepsilon$, $(1.2)_1$ and then let $\eta \to 0$. By condition 2^0, we obtain

$$\int_0^1 |v|dx \le C_1 + C_2\int_0^t\int_0^1 (|v| + |w|)d\tau dx . \tag{3.8}$$

Here and below, c_i denotes a constant independent of ε.

To complete the proof we will make use of condition 1^0. For definiteness, we suppose $f_u < 0$ for $(t,x) \in \bar{\Omega}_T$, $u \in E_0$.

From $(1.1)^{\varepsilon}$,

$$-f_u w = \frac{\partial}{\partial x}[(a+\varepsilon)w] + f_x + g - v\ .$$

Multiplying it by $sgn_\eta(w)$ and integrating over $[0,1]$ with respect to x yield

$$(3.9)\qquad \begin{aligned} -\int_0^1 sgn_\eta(w) f_u w dx &= sgn_\eta(w)(a+\varepsilon)w\Big|_{x=0}^{x=1} - \int_0^1 sgn_\eta'(w)\frac{\partial w}{\partial x}(a+\varepsilon)w dx + \\ &+ \int_0^1 sgn_\eta(w)(f_x+g)dx - \int_0^1 sgn_\eta(w) v dx\ . \end{aligned}$$

Integrating

$$\frac{\partial}{\partial x}[(a+\varepsilon)w + f] = v - g\ ,$$

which is just $(1.1)^{\varepsilon}$, and using $(1.2)_0^{\varepsilon}$, we see that

$$(3.10)\qquad |(a+\varepsilon)w| \le C_3 + \int_0^1 |v|dx\ .$$

Using (3.10) in (3.9) and letting $\eta \to 0$, we obtain

$$(3.11)\qquad -\int_0^1 f_u|w|dx \le C_4 + C_5\int_0^1 |v|dx\ .$$

From condition 1^0 it is easy to see that there exists a constant $\xi > 0$ such that

$$(3.12)\qquad \xi a - f_u > 0 \quad \text{for} \quad (t,x) \in \bar{\Omega}_T,\ |u| \le M\ .$$

Combining (3.11) with

$$\int_0^t\int_0^1 a|w|dt dx \le C_6$$

which follows from (3.5), gives

$$\int_0^t\int_0^1 (\xi a - f_u)|w|dt dx \le C_7 + C_5\int_0^t\int_0^1 |v|d\tau dx\ .$$

Hence, from (3.12) we obtain

$$\int_0^t \int_0^1 |w| d\tau dx \leq C_8 + C_9 \int_0^t \int_0^1 |v| d\tau dx \ .$$

Substitute into (3.8). Then

$$\int_0^1 |v| dx \leq C_{10} + C_{11} \int_0^t \int_0^1 |v| d\tau dx \ .$$

Thus Gronwall's Lemma gives the first estimate of (3.6) and hence the second one of (3.6) follows too. The proof is complete.

Remark. It seems difficult to obtain sharper estimates (like $\int_0^1 \left|\frac{\partial u_\varepsilon}{\partial x}\right| dx \leq M_3$).

Summing up, we obtain

Theorem 3. For the family $\{u_\varepsilon\}$ of solutions of regularized problems $(1.1)^\varepsilon$, $(1.2)_0^\varepsilon$, $(1.2)_1$, (1.3), we have estimates (3.2), (3.5), (3.6).

§4. Existence

In this section, we will further assume that for any $\tau \neq 0$ and $t \in [0,T]$,

$$(4.1) \qquad \int_0^\tau a(t,1,s)ds > 0 .$$

Without this condition, we can proceed in a similar way. However, in that case, a modified definition of generalized solutions should be introduced.

According to Theorem 3, we can conclude that the family $\{u_\varepsilon\}$ is strongly compact in $L^1(\Omega_T)$, namely, there exists $\varepsilon = \varepsilon_n \to 0$ such that $\{u_{\varepsilon_n}\}$ converges both in $L^1(\Omega_T)$ and pointwise a.e. to a function $u \in L^\infty(\Omega_T) \cap BV(\Omega_T)$.

That the limit function u satisfies condition 1) in the definition of generalized solutions is proved just as in [1] and [2].

Let $\phi \in C^\infty(\bar{\Omega}_T)$, $\phi \geq 0$, supp $\phi \subset (0,T) \times (0,1]$. Multiplying $(1.1)^\varepsilon$ by $\mathrm{sgn}_\eta(u_\varepsilon - k)\phi$, integrating over Ω_T and letting $\eta \to 0$ and $\varepsilon = \varepsilon_n \to 0$ successively, we obtain (similar to [2])

$$\iint_{\Omega_T} \mathrm{sgn}(u - k)\Big\{(u - k)\frac{\partial\phi}{\partial t} - (\widehat{a(t},x,u)\frac{\partial u}{\partial x} + f(t,x,u) - f(t,x,k))\frac{\partial\phi}{\partial x} +$$

$$+ (f_x(t,x,k) + g(t,x,u))\phi\Big\}dtdx -$$

$$- \int_0^T \mathrm{sgn}(k)\Big[f(t,x,\gamma u) - f(t,x,k) - \gamma(\widehat{a(t},x,u)\frac{\partial u}{\partial x}\Big]_{x=1} dt +$$

$$+ \int_0^T [\mathrm{sgn}(\gamma u - k) + \mathrm{sgn}(k)][A(t,x,\gamma u) - A(t,x,k)]\frac{\partial\phi}{\partial x}\Big|_{x=1} dt \geq 0$$

where $A(t,x,u) = \int_0^u a(t,x,s)ds$. First, this implies condition 2) in the definition of generalized solutions.

Secondly, as in [2], we can derive

$$[\mathrm{sgn}(\gamma u - k) + \mathrm{sgn}(k)][A(t,x,\gamma u) - A(t,x,k)]_{x=1} = 0 .$$

This and condition (4.1) imply $(1.6)_1$.

The verification of (1.7) is just the same as in [1].

It remains to check $(1.6)_0$. For any $\psi(t) \in C_0^\infty(0,T)$, from $(1.2)_0^\varepsilon$ we have

$$0 = \int_0^T [(a(t,x,u_\varepsilon) + \varepsilon) \frac{\partial u_\varepsilon}{\partial x} + f(t,x,u_\varepsilon)]_{x=0} \psi(t)dt$$

$$= - \int_0^T\int_0^1 \frac{\partial}{\partial x}\{[(a + \varepsilon) \frac{\partial u_\varepsilon}{\partial x} + f] \psi\mu_h\}dtdx$$

$$= - \int_0^T\int_0^1 \frac{\partial}{\partial x} [(a + \varepsilon) \frac{\partial u_\varepsilon}{\partial x} + f] \psi\mu_h dtdx - \int_0^T\int_0^1 [(a + \varepsilon) \frac{\partial u_\varepsilon}{\partial x} + f] \psi\mu_h' dtdx ,$$

where $\mu_h(\sigma) = 1 - \rho_h(\sigma - 2h)$ and $\rho_h(\sigma)$ is the function (2.2).

Notice that

$$- \int_0^T\int_0^1 \frac{\partial}{\partial x} [(a + \varepsilon) \frac{\partial u_\varepsilon}{\partial x} + f] \psi\mu_h dtdx = - \int_0^T\int_0^1 \frac{\partial u_\varepsilon}{\partial t} \psi\mu_h dtdx + \int_0^T\int_0^1 g\psi\mu_h dtdx$$

$$= \int_0^T\int_0^1 u_\varepsilon\psi'\mu_h dtdx + \int_0^T\int_0^1 g\psi\mu_h dtdx$$

$$\xrightarrow[(\varepsilon=\varepsilon_n \to 0)]{} \int_0^T\int_0^1 u\psi'\mu_h dtdx + \int_0^T\int_0^1 g\psi\mu_h' dtdx \xrightarrow[(h\to 0)]{} 0 ,$$

$$- \int_0^T\int_0^1 [(a + \varepsilon) \frac{\partial u_\varepsilon}{\partial x} + f] \psi\mu_h' dtdx = - \int_0^T\int_0^1 \{\frac{\partial(A(t,x,u_\varepsilon) + \varepsilon u_\varepsilon)}{\partial x} -$$

$$- \int_0^{u_\varepsilon} a_x(t,x,s)dx + f(t,x,u_\varepsilon)\}\psi(t)\mu_h'(x)dtdx$$

$$= - \int_0^T (A(t,x,u_\varepsilon) + \varepsilon u_\varepsilon)\psi\mu_h'|_{x=0}^{x=1}dt + \int_0^T\int_0^1 (A(t,x,u_\varepsilon) + \varepsilon u_\varepsilon)\psi\mu_h'' dtdx$$

$$+ \int_0^T\int_0^1\int_0^{u_\varepsilon} a_x(t,x,s)ds\psi\mu_h' dtdx - \int_0^T\int_0^1 f(t,x,u_\varepsilon)\psi\mu_h' dtdx$$

$$\xrightarrow[(\varepsilon=\varepsilon_n \to 0)]{} \int_0^T\int_0^1 A(t,x,u)\psi\mu_h'' dtdx + \int_0^T\int_0^1\int_0^u a_x(t,x,s)ds\psi\mu_h' dtdx$$

$$- \int_0^T\int_0^1 f(t,x,u)\psi\mu_h' dtdx$$

$$= -\int_0^T\int_0^1 [\widehat{a(t},x,u)\,\frac{\partial u}{\partial x} + f(t,x,u)]\,\psi u_h'\,dtdx$$

$$\overrightarrow{(h\to 0)} \int_0^T \gamma(\widehat{a(t},x,u)\,\frac{\partial u}{\partial x} + f(t,x,u))_{x=1}\psi dt\ .$$

Thus for any $\psi \in C_0^\infty(0,T)$, we have

$$\int_0^T \gamma(\widehat{a(t},x,u)\,\frac{\partial u}{\partial x} + f(t,x,u))_{x=1}\psi dt = 0$$

and $(1.6)_0$ follows.

<u>Theorem 4</u> (Existence of generalized solutions). The boundary value problem (1.1), (1.2), (1.3) has a generalized solution u which can be obtained as the limit in $L^1(Q_T)$ of the family $\{u_\varepsilon\}$ of solutions of regularized problems $(1.1)^\varepsilon$, $(1.2)_0^\varepsilon$, $(1.2)_1$, and (1.3).

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It is shown that the arguments developed in Vol'pert and Hudjaev's paper for the Cauchy problem and in the author's paper for the first boundary value problem can be extended to other kinds of boundary value problems. As an example, equations of the form

20. **ABSTRACT** - cont'd.

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(a(t,x,u)\,\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial x} f(t,x,u) + g(t,x,u)$$

$$(a(t,x,u) \geq 0)$$

with the boundary conditions

$$a\,\frac{\partial u}{\partial x} + f = 0 \qquad (x = 0)$$

$$u = 0 \qquad (x = 1)$$

and the initial condition

$$u = u_0(x) \qquad (t = 0)$$

are investigated, and the existence, uniqueness and continuous dependence on the initial value of generalized solutions are proved under certain conditions. In proving the existence, the key step is to establish estimates on solutions u_ε of regularized problem, especially the uniform estimate of $\left|\frac{\partial u_\varepsilon}{\partial t}\right|_{L^1}$ and $\left|\frac{\partial u_\varepsilon}{\partial x}\right|_{L^1}$.